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Criticality of the discrete N-vector ferromagnet in planar self-dual lattices

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Abstract. We study an extended version of the discrete N-vector (or cubic) ferromagnetic model within a real space renormalisation group approach which preserves the two-spin correlation function. The N-evolution (for *real* values of N) of the Wheatstone-bridge hierarchical lattice phase diagram, which presents paramagnetic, intermediate (nematic-like) and ferromagnetic phases, as well as of the thermal (ν) and crossover (ϕ) critical exponents, is presented. The self-avoiding walk problem is recovered in the $N \to 0$ limit, and the so-called 'corner rule' is re-obtained in a larger context. The Ising, N- and $2N$-state Potts ferromagnets are recovered as particular cases. An interchange of stability occurs at $N = N^* \simeq 6.9$ in such a way that the $2N$-state Potts special point (where all three existing phases join) is multicritical if $N < N^*$, but only critical if $N > N^*$ (consistently $\phi(N^*) = 0$). For the cubic model, $\nu(N)$ presents a maximum at $N = N_{max} \simeq 1.5$. The results are exact, for all N, for the Wheatstone-bridge hierarchical lattice, and approximate, for $N \leqslant 2$, for the square lattice. Last but not least, we discuss the connection between the present approach and the phenomenological renormalisation group.

1. Introduction

In recent years several real space renormalisation group (RG) methods have been developed whose transformations describe with reasonable approximation spin models on Bravais lattices, and become exact for the same systems (if classical) on hierarchical structures. The Migdal-Kadanoff approximation as well as the methods developed in [1-4] are examples of this kind of approach. Even if sometimes the approximations involved are not able to reproduce important qualitative features of models on Bravais lattices (like the first-order transitions of a Potts ferromagnet for a sufficiently high number of states [5]), other results can even turn out to be exact, especially when the choice of the basic RG clusters respects some important symmetries of the infinite system (like self-duality for the square lattice). Moreover, as discussed in section 4 of the present paper, if applied to big clusters, the methods of [1-4] can be shown to have the same potentialities as a phenomenological renormalisation approach [6], in which the interfacial tension between different domains in a block is used, in place of the correlation length, as the basic scaling quantity. As illustrations of these potentialities, see [7, 8].

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In this context, particularly appealing is the possibility of treating, within relatively simple renormalisation schemes, whole classes of models, like the q-state Potts model for arbitrary q, or the $Z(N)$ model for arbitrary N. Whereas for the former the renormalisation transformation considered does not require a parameter space with dimension increasing with q, for the most general $Z(N)$ model such dimension grows linearly with N, making the RG quickly intractable. An important consequence of these facts is that, whereas for the Potts model it is possible to have the results for arbitrary *real* values of q (and consequently the important $q \to 1$ and $q \to 0$ limits, respectively the bond percolation and resistor problems, are easily accessible), to obtain for the $Z(N)$ model results which are analytical in N is a non-trivial task.

The main purpose of this paper is to present results, for arbitrary *real* values of N, for a particular realisation of the $Z(2N)$ model, the so-called *discrete N-vector* (or *N-component*) *model* or even *cubic model*. This problem is tractable because, as we shall see, it presents the considerable advantage of requiring, in order to remain closed under renormalisation, a parameter space which is, for any N, at most bidimensional. The cubic model has already been considered within various theoretical frameworks, such as the mean-field approximation [9], Niemejer and van Leeuwen RG [10], Migdal RG [11], variational and dedecoration RG [12], a Monte Carlo-like approach [13], conformal invariance [14] and Monte Carlo RG [15]. Possible physical motivations (e.g. the study of rare-earth compounds) are discussed in [9, 11, 12]. Here we study the cubic model within a RG approach which preserves appropriate two-spin correlation functions. All the results are exact for the Wheatstone-bridge hierarchical lattice; they are either exact (e.g. parts of the phase diagram for $N \leqslant 2$) or approximate (e.g. the critical exponents ν and ϕ) for the square lattice.

In section 2 we introduce the model and the formalism; in section 3 we present the general results as well as those corresponding to the $N \to 0$ limit (self-avoiding walk); in section 4 we make the connection between the present approach and the phenomenological RG; finally we conclude in section 5.

2. Model and formalism

The cubic model elementary interaction between spins i and j is described by the following dimensionless Hamiltonian:

$$\beta \mathcal{H}_{ij} = -NK\boldsymbol{S}_i \cdot \boldsymbol{S}_j \tag{1}$$

where $\beta \equiv 1/k_B T$ and where the spin $\boldsymbol{S}_i$ at any given site is an N-component unitary vector which can point only along the $2N$ positive or negative orthogonal coordinate directions, i.e. $\boldsymbol{S}_i = (\pm 1, 0, 0, \ldots, 0)$ or $(0, \pm 1, 0, \ldots, 0)$ or $\ldots$ or $(0, 0, 0, \ldots, \pm 1)$. This interaction is a discrete version of the classical N-vector model. In what follows we shall consider a generalised form of it, namely

$$\beta \mathcal{H}_{ij} = -NK\boldsymbol{S}_i \cdot \boldsymbol{S}_j - N^2 L(\boldsymbol{S}_i \cdot \boldsymbol{S}_j)^2 \tag{2}$$

which will prove to be closed under the RG.

Hilhorst [10] has verified that model (1) reproduces, in the $N \to 0$ limit for $L = 0$, the grand-canonical statistics of a self-avoiding walk (SAW) with step fugacity K. This result also holds for model (2) and extends to discrete spins the de Gennes result [16] for continuous spins; it was in fact exploited for the early RG analysis of the SAW mentioned above [10]. For the particular case $N = 1$, model (2) reduces to the spin-$\frac{1}{2}$

Ising model for all values of L. For $N=2$ we recover the Z(4) model (see, for example, [4] and references therein). If $NL=K$, model (2) recovers the $2N$-state Potts model with dimensionless coupling constant $2NK$. If $K=0$, model (2) recovers the N-state Potts model with dimensionless coupling constant N^2L. For finite K and $NL/|K|\to\infty$ we recover, for all values of N, the spin-$\frac{1}{2}$ Ising model with dimensionless coupling constant NK. Indeed, the second term of Hamiltonian (2) becomes dominant, and therefore only parallel and antiparallel spin configurations are possible at any finite temperature. To summarise all these particular situations, let us say, by using the notation of the (N_α, N_β) model introduced by Domany and Riedel [11], that Hamiltonian (2) corresponds to the $(N, 2)$ model.

Hamiltonian (2) is in general associated with a three-level system. For instance, if we assume $K > NL > 0$, we have a fundamental level whose energy is $-N(K+NL)$ and whose degeneracy is $2N$; the energy of the first excited level is 0 and its degeneracy is $4N(N-1)$; finally, the energy of the second excited level is $N(K-NL)$ and its degeneracy is $2N$.

If we consider now a two-rooted graph made by a series array of two bonds with coupling constants $(K^{(1)}, L^{(1)})$ and $(K^{(2)}, L^{(2)})$ respectively, its Hamiltonian will be given by

$$\beta\mathcal{H}_{123} = -NK^{(1)}\boldsymbol{S}_1\cdot\boldsymbol{S}_3 - N^2L^{(1)}(\boldsymbol{S}_1\cdot\boldsymbol{S}_3)^2 - NK^{(2)}\boldsymbol{S}_3\cdot\boldsymbol{S}_2 - N^2L^{(2)}(\boldsymbol{S}_3\cdot\boldsymbol{S}_2)^2 \tag{3}$$

where $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are the terminal spins and $\boldsymbol{S}_3$ the internal one. For all statistical equilibrium properties which do not directly involve $\boldsymbol{S}_3$, $\beta\mathcal{H}_{123}$ can be replaced by

$$\beta\mathcal{H}'_{12} = -NK^{(s)}\boldsymbol{S}_1\cdot\boldsymbol{S}_2 - N^2L^{(s)}(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)^2 - K'_0 \tag{4}$$

(s stands for *series*) where we impose

$$\exp(-\beta\mathcal{H}'_{12}) = \mathrm{Tr}_3\exp(-\beta\mathcal{H}_{123}) \tag{5}$$

with $K^{(s)}$, $L^{(s)}$ and K'_0 to be determined. The results (except for K'_0, which is not important in the present context) can be written as follows:

$$t_r^{(s)} = t_r^{(1)}t_r^{(2)} \qquad r=1,2 \tag{6}$$

where the *vector thermal transmissivity* (t_1, t_2) (see [2, 4, 12]) is related to (K, L) through the definitions

$$t_1 \equiv \frac{1-\exp(-2NK)}{1+2(N-1)\exp[-N(K+NL)]+\exp(-2NK)} \tag{7a}$$

and

$$t_2 \equiv \frac{1-2\exp[-N(K+NL)]+\exp(-2NK)}{1+2(N-1)\exp[-N(K+NL)]+\exp(-2NK)}. \tag{7b}$$

For the $2N$-state Potts model ($K=NL$) we have $t_1=t_2$, for the N-state Potts model ($K=0$) we have $t_1=0$, and for the Ising model ($NL/|K|\to\infty$) we have $t_2=1$. In all these cases we recover the definition of thermal transmissivity introduced in [2]. For $N=2$, (t_1, t_2) reproduces the vector transmissivity of the Z(4) model as defined in [4]. It is finally worth mentioning that the cubic model ($L=0$) corresponds to the equation $(N-2)t_2^2+2t_2=Nt_1^2$.

Equations (7) yield, through inversion,

$$\exp[-N(K+NL)] = \frac{1-t_2}{1+Nt_1+(N-1)t_2} \tag{8a}$$

and

$$\exp(-2NK)=\frac{1-Nt_1+(N-1)t_2}{1+Nt_1+(N-1)t_2}. \tag{8b}$$

We note that for $N=2$ and only then, the functional forms of the transformation $(t_1, t_2)\rightleftarrows(\exp[-N(K+NL)], \exp(-2NK))$ are one and the same. In other words, if we define $(t_1, t_2)=F_N(\exp[-N(K+NL)], \exp(-2NK))$, in general $F_N^{-1}\neq F_N$, but $F_2^{-1}=F_2$. This fact will make, as we shall see further on, a special case of the $N=2$ model.

Let us now consider a *parallel* (instead of series) array of two bonds with coupling constants $(K^{(1)}, L^{(1)})$ and $(K^{(2)}, L^{(2)})$. The equivalent coupling constants $(K^{(p)}, L^{(p)})$ will be now given by

$$K^{(p)}=K^{(1)}+K^{(2)} \tag{9a}$$

and

$$L^{(p)}=L^{(1)}+L^{(2)} \tag{9b}$$

or equivalently

$$t_1^{(p)}=\frac{t_1^{(1)}+t_1^{(2)}+(N-1)t_2^{(1)}t_1^{(2)}+(N-1)t_1^{(1)}t_2^{(2)}}{1+Nt_1^{(1)}t_1^{(2)}+(N-1)t_2^{(1)}t_2^{(2)}} \tag{10a}$$

$$t_2^{(p)}=\frac{t_2^{(1)}+t_2^{(2)}+Nt_1^{(1)}t_1^{(2)}+(N-2)t_2^{(1)}t_2^{(2)}}{1+Nt_1^{(1)}t_1^{(2)}+(N-1)t_2^{(1)}t_2^{(2)}}. \tag{10b}$$

These equations can also be written as follows:

$$(t_r^{(p)})^{\mathrm{D}}=(t_r^{(1)})^{\mathrm{D}}(t_r^{(2)})^{\mathrm{D}} \qquad r=1,2$$

with

$$t_1^{\mathrm{D}}\equiv\frac{1-Nt_1+(N-1)t_2}{1+Nt_1+(N-1)t_2} \tag{11a}$$

$$t_2^{\mathrm{D}}\equiv\frac{1-t_2}{1+Nt_1+(N-1)t_2}. \tag{11b}$$

For a full discussion of this kind of 'dual' variable see [17].

Now that we have introduced the variables t_1 and t_2 (very convenient at the present time for representing the RG flow diagrams, and possibly in future for formulating a break-collapse method [2, 4, 17, 18]), let us focus on the ferromagnetic model on the square lattice. The Hamiltonian will be given by

$$\beta\mathscr{H}=-NK\sum_{\langle i,j\rangle}(\boldsymbol{S}_i\cdot\boldsymbol{S}_j)-N^2L\sum_{\langle i,j\rangle}(\boldsymbol{S}_i\cdot\boldsymbol{S}_j)^2 \tag{12}$$

where the sums run over all pairs of nearest-neighbour sites, $K>0$ and $L\geqslant -K/N$.

In a way similar to what happens for the Potts ferromagnet, the $L=0$ transition is expected to become first order on a Bravais lattice for large enough N. Mean-field theory predicts a first-order transition for $N>3$ [9]. Real space renormalisation group calculations in two dimensions indicated a first-order transition for $N>N_c=2$ [15]. At the present moment we will leave out the discussion of the aspects connected with the first-order transition, and focus more on the peculiar features of the phase diagram of hierarchical lattices, which can be obtained exactly without the introduction of

vacancies [12]. The hierarchical lattice we consider here in particular is that corresponding to the Wheatstone bridge cluster of figure 1. This cluster, due to its self-duality, guarantees coincidence of the critical couplings with those of the infinite square lattice, in all cases in which the model becomes self-dual.

To construct our RG we impose

$$\exp(-\beta\mathscr{H}'_{12}) = \operatorname*{Tr}_{3,4} \exp(-\beta\mathscr{H}_{1234}) \tag{13}$$

where $\mathscr{H}'_{12}$ and $\mathscr{H}_{1234}$ respectively are the Hamiltonians associated with the small and large graphs of figure 1 ($\mathscr{H}'_{12}$ in particular is explicitly written in (4) with (K', L') replacing $(K^{(s)}, L^{(s)})$). Equation (13) yields

$$K' = \frac{1}{2N} \ln \frac{G_1}{G_2} \tag{14}$$

and

$$L' = \frac{1}{2N^2} \ln \frac{G_1 g_2}{G_3^2} \tag{15}$$

with

$$\begin{aligned} G_1 \equiv{}& \exp(5N^2L)[\exp(5NK) + \exp(-3NK) + 2\exp(-NK)] \\ &+ 2(N-1)\{2\exp(2N^2L)[\exp(2NK) + \exp(-2NK)] \\ &+ \exp(N^2L)[\exp(NK) + \exp(-NK)] + 2N - 4\} \end{aligned} \tag{16}$$

$$\begin{aligned} G_2 \equiv{}& 2\exp(5N^2L)[\exp(NK) + \exp(-NK)] + 2(N-1))\{4\exp(2N^2L) \\ &+ \exp(N^2L)[\exp(NK) + \exp(-NK)] + 2N - 4\} \end{aligned} \tag{17}$$

$$\begin{aligned} G_3 \equiv{}& 2[\![\exp(3N^2L)[\exp(3NK) + 3\exp(-NK)] + \exp(2N^2L) \\ &\times [\exp(2NK) + 2 + \exp(-2NK)] \\ &+ (N-2)\{5\exp(N^2L)[\exp(NK) + \exp(-NK)] + 2N - 6\}]\!]. \end{aligned} \tag{18}$$

Equations (14) and (15) provide the RG recurrence relations we were looking for. For fixed N, the RG flow in the (K, L) space (or equivalently in the (t_1, t_2) space) will determine the phase diagram as well as the universality classes. The numerical values of thermal and crossover exponents (ν and ϕ respectively) can be obtained through the calculation of the Jacobian matrix $\partial(K', L')/\partial(K, L)$ on the various semistable or fully unstable fixed points. More specifically, if we denote by λ_1 and λ_2 the eigenvalues of the matrix we have the following.

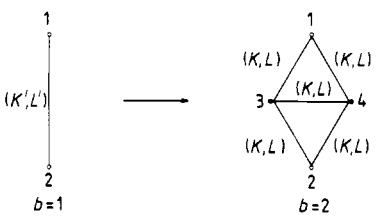

Figure 1. Iteration associated with the Wheatstone-bridge RG (the full circles and the open circles respectively denote the internal and terminal sites of the graph).

(i) $\lambda_1 > 1 > \lambda_2$ for critical (semistable) fixed points, and

$$\nu = \frac{\ln B}{\ln \lambda_1} \tag{19}$$

where B is the linear expansion factor ($B = 2$ for figure 1).

(ii) $\lambda_1 > 1$ and $\lambda_2 > 1$ for multicritical (fully unstable) fixed points,

$$\nu_s = \frac{\ln B}{\ln \lambda_s} \qquad s = 1, 2 \tag{20}$$

and

$$\phi = \frac{\ln \lambda_2}{\ln \lambda_1} \tag{21}$$

where λ_2 denotes that eigenvalue which, as N varies, tends to unity whereas λ_1 remains greater than unity.

3. General results

The phase diagrams for typical values of N are presented in figures 2(a) (in the (t_1, t_2) variables) and 2(b) (in the $(1/K, NL/K)$ variables). For a given value of N, the phase diagram presents three phases, namely the *paramagnetic* (P; characterised by the fully stable fixed point $t_1 = t_2 = 0$), the *ferromagnetic* (F; characterised by the fully stable fixed point $t_1 = t_2 = 1$) and the *intermediate* (I; characterised by the fully stable fixed point $(t_1, t_2) = (0, 1)$) phases. The existence of three distinct phases is well known for $N = 2$ (Z(4) model). This structure analytically remains so for *all* values of N, including for $N \leq 1$ where it should be considered as a mathematical artefact. Indeed, for $N = 1$ (Ising model), the P-I critical frontier should be considered as spurious, since for this

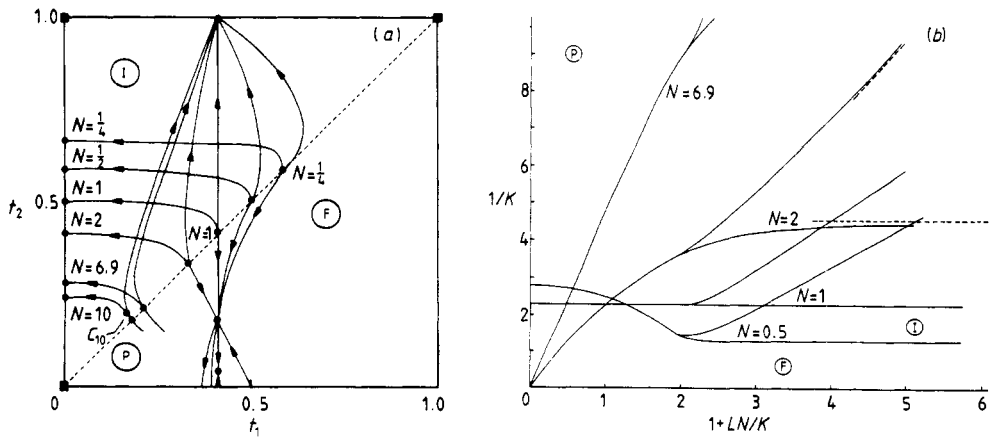

Figure 2. Phase diagram in the (t_1, t_2) space for typical values of N: P, F and I respectively denote the paramagnetic, ferromagnetic and intermediate phases. The arrows indicate the RG flow; the full squares and the full circles respectively indicate stable and unstable fixed points. The line $t_1 = t_2$ corresponds to the $2N$-state Potts model. (b) Phase diagram in the $(1/K, 1 + LN/K)$ space for typical values of N.

model only two distinct phases exist, namely the ferromagnetic phase (F) and the paramagnetic phase (P and I); as expected, the physically meaningful critical temperature for $N = 1$, does not depend on NL/K ('vertical' line in figure 2(a), and 'horizontal' line in figure 2(b)).

The critical frontier corresponding to a given value of N contains *four* special points, namely three semistable fixed points (critical points) and a fully unstable point (multicritical point). Two of the three critical points are the Ising critical point $((t_1, t_2) = (\sqrt{2}-1, 1))$ and the N-state Potts critical point $((t_1, t_2) = (0, 1/(\sqrt{N}+1))$. The third and fourth special points are the $2N$-state Potts special point $(t_1 = t_2 = 1/(\sqrt{2N}+1))$ and the *extended cubic* special point $((t_1, t_2) = (t_1^c, t_2^c)$ where the associated transmissivities and coupling constants are given in figures 3(a) and 3(b) respectively). For $N < N^* \simeq 6.9$ the $2N$-state Potts model corresponds to the multicritical point and the extended cubic model corresponds to the critical point; the situation is reversed for $N > N^*$. At $N = N^*$ a special multicritical point emerges as the $2N$-state Potts and the extended cubic fixed points collapse; at this value of N the two models exchange stability.

The thermal critical exponent ν_T as well as the crossover exponent ϕ are shown in figures 4(a) and 4(b) for the $2N$-state Potts and the extended cubic models

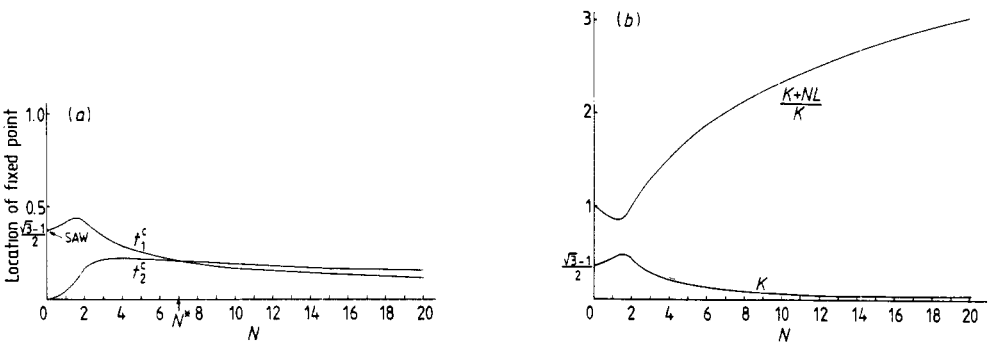

Figure 3. N dependence of the location of the extended cubic fixed point: (a) (t_1, t_2) variables; (b) $(K, 1+NL/K)$ variables.

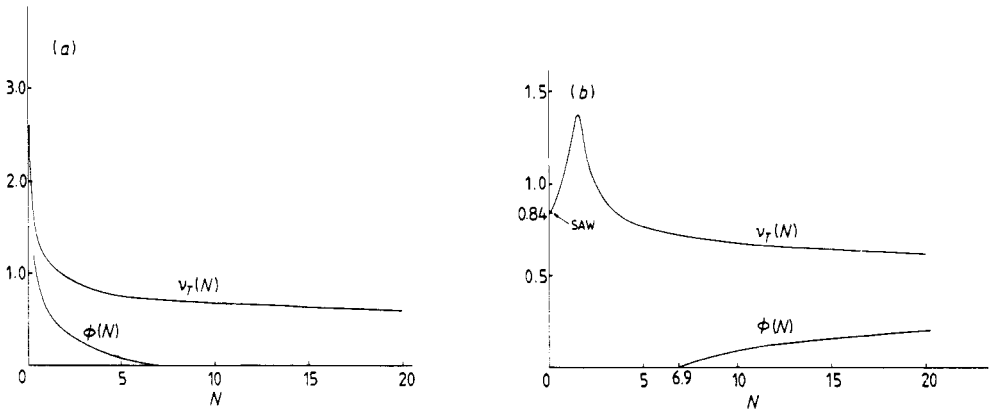

Figure 4. N dependence of the thermal critical exponent ν_T and the crossover exponent ϕ: (a) $2N$-state Potts model; (b) extended cubic model.

respectively. In particular, in figure 4(a) we recover the well known values of ν_T for the Wheatstone-bridge hierarchical lattice Potts model, namely $\nu_T \simeq 1.43$ for the bond percolation model ($N = \frac{1}{2}$), and $\nu_T \simeq 1.15$ for the Ising model ($N = 1$); it is also worth mentioning that $\phi = 1$ for $N = \frac{1}{2}$. Also, in the $N \to \infty$ limit we obtain $\nu_T = \ln 2/\ln 5 \simeq 0.43$, in accordance with the conjecture [19, 20] that ν_T should give $1/d_f$ where d_f is the intrinsic fractal dimensionality. Finally, our numerical results suggest that, in the limit $N \to \infty$, the exponents ν_T associated with the $2N$-Potts and extended cubic models coincide.

A limit of special interest is the $N \to 0$ limit, as it corresponds to the self-avoiding walk problem (SAW). In the figure 3(b) we see that $K_c = (\sqrt{3}-1)/2 \simeq 0.366$ which corresponds to the exact critical fugacity for the Wheatstone-bridge hierarchical lattice (for the square lattice we have $K_c \simeq 0.3790$ [21]). The corresponding value for ν_T is given by $\nu_T = \ln 2/\ln(4-\sqrt{3}) \simeq 0.85$ (see figure 4(b)), to be compared with the value $\frac{3}{4}$ [21]. In fact, the present RG precisely recovers (and consequently further supports), in the $N \to 0$ limit, the 'corner rule' [22]. Indeed, this rule provides the RG recursive relation $K' = 2K^2 + 2K^3$, whose critical fixed point and thermal exponent are precisely $K_c = (\sqrt{3}-1)/2$ and $\nu_T = \ln 2/\ln(4-\sqrt{3})$.

4. Connection with the phenomenological RG approach

As stressed in the previous sections, the renormalisation procedure applied in this work [1-4] is exact for a hierarchical lattice, while it is expected to be a more or less good approximation for systems on a Bravais lattice. In this section we intend to better clarify the nature of this approximation by making explicit the connection between the present approach and the phenomenological RG [6] (see also [23]).

To avoid unnecessary complications, let us focus on the particular case of the $d = 2$ Ising model ($N = 1$). We can omit vector notation and represent the spin at site i simply by $S_i = \pm 1$.

Successive clusters of the Wheatstone-bridge family are reported in figure 5 (the $b = 1$ and $b = 2$ clusters are shown in figure 1). On each of these clusters (with $b(b-1)$ internal spins), the summation procedure leading to the renormalised coupling constant K' can be interpreted as the calculation of an *interface* free energy for blocks of the type indicated in figure 6. The spins on the upper and lower horizontal sides of the block are left out of the summation. Indeed, if we indicate by $\{S\}$ the configurations of the internal spins of the cluster (i.e. other than S_1 and S_2) we have:

$$\exp(K' S_1 S_2 + g) = \operatorname*{Tr}_{\{S\}} \exp(-\beta \mathcal{H}(\{S\}; S_1, S_2)) \equiv Z_{S_1 S_2}(K) \tag{22}$$

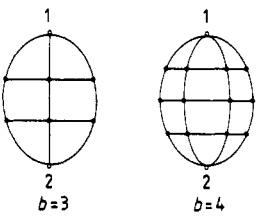

Figure 5. $b = 3$ and $b = 4$ generating graphs of the Wheatstone-bridge family of hierarchical lattices.

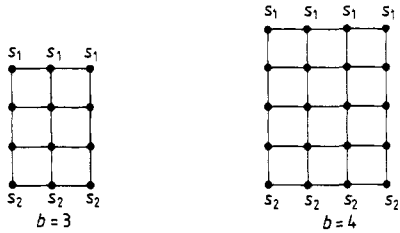

Figure 6. $b=3$ and $b=4$ blocks of spins respectively corresponding to those of figure 5.

where g is an appropriate spin-independent term. From (22) we obtain

$$K' \equiv K'(K, b) = \tfrac{1}{2}[\ln Z_{++} - \ln Z_{+-}]. \tag{23}$$

This means that K' is nothing but the dimensionless excess free energy produced by fixing the horizontal sides to (+) and (−), compared with the case in which both sides are fixed, say, to (+). By definition of the (dimensionless) surface tension σ, we thus have

$$K'(K, b) = (b-1)\sigma(K, b) \tag{24}$$

where $\sigma(K, b)$ is expected to become independent of b in the $b \to \infty$ limit (thermodynamic limit).

From finite-size scaling [24] we expect, for $K \sim K_c$ and $b \to \infty$,

$$\sigma(K, b) \sim b^{-1}\sigma_0(b/\xi_\infty(K)) \sim 1/\xi(K, b) \tag{25}$$

where $\xi_\infty(K)$ is the correlation length of the infinite system, σ_0 is a scaling function with $\sigma_0(0) \neq 0$, and $\xi(K, b)$ is the correlation length in the finite block.

If we now define, as is often done [1, 2, 7], a renormalised coupling constant K_{ren} corresponding to a linear rescaling factor b/b' $(b' < b)$, through the following cell to cell recurrence relation:

$$K'(K_{ren}, b') = K'(K, b) \tag{26}$$

it follows, from (24) and (25) and for large b and b', that

$$\xi(K_{ren}, b') = \frac{b'}{b}\,\xi(K, b). \tag{27}$$

This is nothing but the definition of renormalised coupling constant in a phenomenological approach [6]. It is clear that various choices can be made for the cells to be used. In particular, the standard choice in the phenomenological approach is finite × infinite strips, whereas here we are using finite × finite self-dual clusters. In view of the nice convergence of results generally obtained with phenomenological renormalisation methods, the preceding arguments justify using the usual strategy for improving the results obtained herein (as well as in similar treatments) i.e., that of considering cell to cell transformations $K \to K_{ren}$, as in (26), with both b and b' becoming increasingly large (as is usually done in the phenomenological RG).

The above derivation can of course be easily generalised to the case of dimensionality $d \neq 2$, and to models other than the Ising model.

Summarising, we see that the procedure we have used here should not be interpreted as another type of decimation RG approximation. Indeed, although we impose the correlation function to be preserved, we do so between the roots of the graphs, which

corresponds to imposing the surface free energy to be preserved in the Bravais blocks, whereas in the decimation procedures what is imposed is the preservation of the correlation function between two sites of the Bravais lattice. This makes a substantial difference since the decimation procedures, unless conveniently handled, introduce intrinsic difficulties related to the spin rescaling. These difficulties do not exist in the present approach.

The present analysis makes it clear that the well known limitations of the Migdal-Kadanoff-like approaches are not due to the fact that correlation functions are preserved, but rather to the fact that diamond (or trees) choices for the graphs lead, even for large clusters, to topologies which are not at all those of the Bravais lattices which are supposed to be approached.

5. Conclusion

We have considered the criticality of the discrete N-vector ferromagnet in planar self-dual lattices. The real space renormalisation group approach we used exactly preserves correlation functions between the roots of conveniently chosen two-rooted graphs. The renormalisation leaves invariant not the standard discrete N-vector model (cubic model) but a generalised version thereof. The results are exact for the associated hierarchical lattices, and good estimates for the square lattice ($N \leqslant 2$). The phase diagram (including multicritical points) associated with fixed N, as well as the thermal and crossover exponents, are calculated. At a particular value of N (denoted N^*) an exchange of stability is observed between the Potts and cubic models ($N^* \simeq 6.9$ for the Wheatstone-bridge hierarchical lattice). In the $N \to 0$ limit we recover the self-avoiding walk, and give support to the 'corner rule' which has long been used in this problem.

In addition to the above results, we have exhibited the connection between the present (correlation-function preserving) renormalisation procedure and the phenomenological renormalisation group. This connection makes clear that these two commonly used renormalisation procedures share essentially the same advantages and limitations.

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